Problem 1. Let n be a positive integer. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$(f(x))^n f(x+y) = (f(x))^{n+1} + x^n f(y)$$

for all $x, y \in \mathbb{R}$.

Solution. The functions we are looking for are $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ and $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$. For n even $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x$ is also a solution.

Throughout the solution, $P(x_0, y_0)$ will denote the substitution of x_0 and y_0 for x and y, respectively, in the given equation.

$P(x, 0)$ for $x \neq 0$ gives

$$f(x)^{n+1} = f(x)^{n+1} + x^n f(0)$$

and therefore

$$f(0) = \frac{f(x)^{n+1} - f(x)^{n+1}}{x^n} = 0.$$

$P(x, -x)$ for $x \neq 0$ gives

$$0 = f(x)^n f(0) = f(x)^{n+1} + x^n f(-x),$$

and therefore

$$f(-x) = -\frac{f(x)^{n+1}}{x^n}.$$

Applying this identity twice, we get

$$f(x) = f(-(-x)) = -\frac{f(-x)^{n+1}}{(-x)^n} = -\frac{\left(-\frac{f(x)^{n+1}}{x^n}\right)^{n+1}}{(-x)^n} = \frac{f(x)^{n^2+2n+1}}{x^{n^2+2n}},$$

which after rearranging yields

$$f(x)(x^{n^2+2n} - f(x)^{n^2+2n}) = 0.$$

If there exists an $a \neq 0$ for which $f(a) = 0$, then $P(a, y)$ yields

$$0 = a^n f(y),$$

which means that $f(y) = 0$ for all $y \in \mathbb{R}$. This is a solution to the equation for all n.

If instead $f(x) \neq 0$ for all $x \neq 0$, then we have

$$x^{n^2+2n} = f(x)^{n^2+2n}.$$

If n is odd, then so is $n(n+2) = (n^2 + 2n)$, meaning $f(x) = x$ for all $x \in \mathbb{R}$. This is a solution to the equation.

If n is even, then so is $n(n+2) = (n^2+2n)$, meaning $f(x) = \pm x$ for all $x \in \mathbb{R}$. Both $f(x) = x$ and $f(x) = -x$ are solutions to the equation. In all other cases there must exist $x, y \neq 0$ such that $f(x) = x$ and $f(y) = -y$. Then $P(x, y)$ yields

$$x^n f(x+y) = x^{n+1} - x^n y,$$

which after dividing by $x^n \neq 0$ yields

$$f(x+y) = x - y.$$

Since $(f(x))^2 = x^2$ for all $x \in \mathbb{R}$, we have $(x+y)^2 = (x-y)^2$. That is $4xy = 0$ which is impossible as $x, y \neq 0$.

There are therefore no more solutions to the equation. □

Problem 2. Let a, b, c be the side lengths of a triangle. Prove that

$$\sqrt[3]{(a^2+bc)(b^2+ca)(c^2+ab)} > \frac{a^2+b^2+c^2}{2}.$$

Solution. We claim that

$$a^2 + bc > \frac{a^2+b^2+c^2}{2},$$

which will finish the proof. Note that the claimed inequality is equivalent to

$$\begin{aligned} a^2 + bc > \frac{a^2+b^2+c^2}{2} &\iff 2a^2 + 2bc > a^2 + b^2 + c^2 \\ &\iff a^2 > (b-c)^2 \iff a > |b-c|, \end{aligned}$$

which holds due to the assumption of a, b, c being side lengths of a triangle. □

Problem 3. Determine all infinite sequences $(a_1, a_2, \dots)$ of positive integers satisfying

$$a_{n+1}^2 = 1 + (n+2021)a_n$$

for all $n \geq 1$.

Solution. Clearly $(a_n)_{n=1}^{\infty} = (n + 2019)_{n=1}^{\infty}$ is a solution. We claim that it is the only one.

Assume $(a_n)_{n=1}^{\infty}$ is a solution. Let $(b_n)_{n=1}^{\infty} = (a_n - n)_{n=1}^{\infty}$. We claim:

1. If $b_n < 2019$, then $2019 > b_{n+1} > b_n$.
2. If $b_n > 2019$, then $2019 < b_{n+1} < b_n$.

It is clear that these claims imply that $b_n = 2019$ for all n.
Let us prove the claims:

1. Clearly $a_n < n + 2019$ implies that $a_{n+1} < (n + 1) + 2019$ which already proves one part of the inequality. Suppose that $b_{n+1} \le b_n$. Then
$$(n + 1 + b_n)^2 \ge a_{n+1}^2 = 1 + (n + 2021)(n + b_n).$$
Expanding gives
$$(n + b_n)(b_n - 2019) \ge 0$$
which shows that $b_n \ge 2019$ contradicting our assumption.
2. It follows in exactly the same way, by just reversing all the inequality signs.

□

Problem 4. Let Γ be a circle in the plane and S be a point on Γ. Mario and Luigi drive around the circle Γ with their go-karts. They both start at S at the same time. They both drive for exactly 6 minutes at constant speed counterclockwise around the track. During these 6 minutes, Luigi makes exactly one lap around Γ while Mario, who is three times as fast, makes three laps.

While Mario and Luigi drive their go-karts, Princess Daisy positions herself such that she is always exactly in the middle of the chord between them. When she reaches a point she has already visited, she marks it with a banana.

How many points in the plane, apart from S, are marked with a banana by the end of the race?

Solution 1. Without loss of generality, we assume that Γ is the unit circle and $S = (1, 0)$. Three points are marked with bananas:

(i) After 45 seconds, Luigi has passed through an arc with a subtended angle of 45° and is at the point $(\sqrt{2}/2, \sqrt{2}/2)$, whereas Mario has passed through an arc with a subtended angle of 135° and is at the point $(-\sqrt{2}/2, \sqrt{2}/2)$. Therefore Daisy is at the point $(0, \sqrt{2}/2)$ after 45 seconds. After 135 seconds, Mario and Luigi's positions are exactly the other way round, so the princess is again at the point $(0, \sqrt{2}/2)$ and puts a banana there.

(ii) Similarly, after 225 seconds and after 315 seconds, Princess Daisy is at the point $(0, -\sqrt{2}/2)$ and puts a banana there.

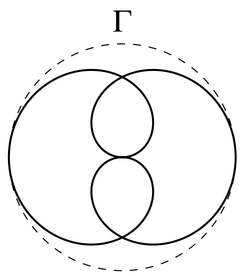
(iii) After 90 seconds, Luigi is at $(0, 1)$ and Mario at $(0, -1)$, so that Daisy is at the origin of the plane. After 270 seconds, Mario and Luigi's positions are exactly the other way round, hence Princess Daisy drops a banana at the point $(0, 0)$.

We claim that no other point in the plane, apart from these three points and S, is marked with a banana. Let t_1 and t_2 be two different times when Daisy is at the same place. For $n \in \{1, 2\}$ we write Luigis position at time t_n as a complex number $z_n = \exp(ix_n)$ with $x_n \in]0, 2\pi[$. At this time, Mario is located at z_i^3 and Daisy at $(z_i^3 + z_i)/2$. According to our assumption we have $(z_1^3 + z_1)/2 = (z_2^3 + z_2)/2$ or, equivalently, $(z_1 - z_2)(z_1^2 + z_1 z_2 + z_2^2 + 1) = 0$. We have $z_1 \neq z_2$, so that we must have $z_1^2 + z_1 z_2 + z_2^2 = -1$.

We proceed with an observation of the structure of Γ as a set of complex numbers. Suppose that $z \in \Gamma \setminus \{S\}$. Then $z + 1 + z^{-1} \in \Gamma$ if and only if $z \in \{i, -1, -i\}$. For a proof of the observation note that $z + 1 + z^{-1} = z + 1 + \overline{z}$ is a real number for every $z \in \mathbb{C}$ with norm $|z| = 1$. So it lies on the unit circle if and only if it is equal to 1, in which case the real part of z is equal to 0, or it is equal to -1, in which case the real part of z is equal to -1. We apply the observation to the number $z = z_1/z_2$, which satisfies the premise since $z + 1 + \overline{z} = -\overline{z_1} \cdot \overline{z_2} \in \Gamma$. Therefore, one of the following cases must occur.

(i) We have $z = \pm i$, that is, $z_1 = \pm i z_2$. Without loss of generality we may assume $z_2 = i z_1$. It follows that $-1 = z_1^2 + z_1 z_2 + z_2^2 = i z_1^2$, so that $z_1 = \exp(i\pi/4)$ or $z_1 = \exp(5i\pi/4)$. In the former case $(z_1, z_2) = (\exp(i\pi/4), \exp(3i\pi/4))$, which matches case (1) above. In the latter case $(z_1, z_2) = (\exp(5i\pi/4), \exp(7i\pi/4))$, which matches case (2) above.

(ii) We have $z = -1$, that is, $z_2 = -z_1$. It follows that $-1 = z_1^2 + z_1 z_2 + z_2^2 = z_1^2$, so that $z_1 = i$ or $z_1 = -i$. This matches case (3) above.



Depiction of the path Daisy takes

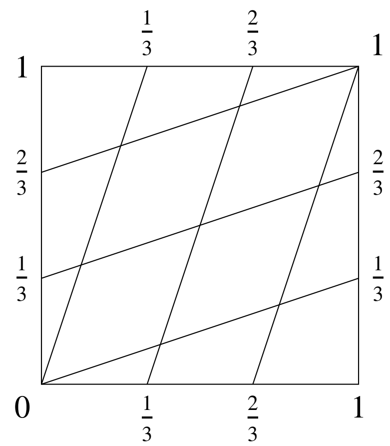
$\square$

Solution 2. We represent the position of Luigi and Mario by $(t, 3t) \pmod 1$, so when Luigi is at angle $t \cdot 2\pi$, Mario is at an angle $3t \cdot 2\pi$.

A chord (if it is not the diameter is determined by its midpoint. Therefore Daisy revisit a location only if

$$(x, 3x) \equiv (3y, y) \pmod 1$$

for different value of $x, y \pmod 1$. By inspection, the three locations are visited twice.



□

Problem 5. Let $x, y \in \mathbb{R}$ be such that $x = y(3-y)^2$ and $y = x(3-x)^2$. Find all possible values of $x + y$.

Solution 1. The set $\{0, 3, 4, 5, 8\}$ contains all possible values for $x + y$.

A pair $(x, x) \in \mathbb{R}^2$ satisfies the equations if and only if $x = x(3-x)^2$, and it is easy to see that this cubic equation has the solution set $\{0, 2, 4\}$. These pairs give us 0, 4 and 8 as possible values for $x + y$.

Assuming $x \neq y$ let s be the sum $x + y$ and p be the product xy. Subtracting the first equation from the second and cancelling out the term $x - y$ we get

$$p = s^2 - 6s + 10.$$

Adding the two equations gives

$$0 = s(s^2 - 3p) - 6(s^2 - 2p) + 8s.$$

Together the equations give

$$0 = s^3 - 12s^2 + 47s - 60 = (s-3)(s-4)(s-5).$$

The only possible values for $x + y$ when $x \neq y$ are therefore $3, 4$ and 5. We have already seen that $x + y = 4$ has a solution $x = y = 2$.

Next we investigate the case $x + y = 3$. Here we can simplify the given equations as $x = yx^2$ and $y = xy^2$. The number x cannot be zero in this case, since otherwise y and k would also be zero. We can conclude that $xy = 1$. The equations $x + y = 3$ and $xy = 1$, according to Vieta's Theorem, imply that x and y are the solutions of the equation $\lambda^2 - 3\lambda + 1 = 0$. Hence

$$(x, y) = \left(\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right) \quad \text{or} \quad (x, y) = \left(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right)$$

and it is easy to verify that both pairs satisfy the equations. These pairs give us 3 as a possible value for $x + y$.

A simple calculation shows that if a pair $(x, y) \in \mathbb{R}^2$ satisfy the equations, then the pair $(4-x, 4-y)$ is solution to the equations. From the pairs we have just found, we can therefore construct pairs of solutions

$$(x, y) = \left(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right) \quad \text{and} \quad (x, y) = \left(\frac{5+\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right),$$

which give us 5 as a possible value for $x + y$.

$\square$

Solution 2. Let $f(x) = x(3 - x)^2$. It is easy to check that if $x < 0$ then $f(x) < x$. In particular $f(f(x)) < f(x) < x$ in this case, so that the pair $(x, f(x))$ cannot be a solution. Similarly, $f(x) > x$ if $x > 4$, so the pair $(x, f(x))$ cannot be a solution in this case either.

Suppose that $(x, y) \in \mathbb{R}^2$ is a solution. According to the previous remark $x \in [0, 4]$, and similarly, $y \in [0, 4]$. Hence we may write $x = 2 + 2r$ and $y = 2 + 2s$ with $r, s \in [-1, 1]$. After substitution and simplification, the equation $x = y(3 - y)^2$ transforms into the equation $r = 4s^3 - 3s$. Recall the trigonometric identities for threefold angles. If $s = \cos(\alpha)$ for some $\alpha \in \mathbb{R}$, then $r = 4\cos^3(\alpha) - 3\cos(\alpha) = \cos(3\alpha)$. In the same way $s = 4r^3 - 3r = \cos(9\alpha)$.

We can deduce that $9\alpha = 2\pi m + \alpha$ or $9\alpha = 2\pi l - \alpha$ for some integers m and l. In the former case we have $8\alpha = 2\pi m$, so that $m \in \{0, 1, 2, 3, 4\}$, and the corresponding possible pairs of solutions can be found in Figure 1. In the former case we have $10\alpha = 2\pi l$, so that $l \in \{0, 1, 2, 3, 4, 5\}$, where $l = 0$ and $l = 5$ result in angles that we have already considered in the first case. We consider the other options in Figure 2 taking into account the well-known identities $\cos(\pi/5) = (1 + \sqrt{5})/4$ and $\cos(3\pi/5) = (1 - \sqrt{5})/4$.

$\square$

m	8α	α	r	s	x	y	$x+y$
0	0	0	1	1	4	4	8
1	2π	$\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	$2+\sqrt{2}$	$2-\sqrt{2}$	4
2	4π	$\pi/2$	0	0	2	2	4
3	6π	$3\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	$2-\sqrt{2}$	$2+\sqrt{2}$	4
4	8π	π	-1	-1	0	0	0

Figure 1: Pairs of solutions and their sums

l	10α	α	r	s	x	y	$x+y$
1	2π	$\pi/5$	$(1+\sqrt{5})/4$	$(1-\sqrt{5})/4$	$(5+\sqrt{5})/2$	$(5-\sqrt{5})/2$	5
2	4π	$2\pi/5$	$(-1+\sqrt{5})/4$	$(-1-\sqrt{5})/4$	$(3+\sqrt{5})/2$	$(3-\sqrt{5})/2$	3
3	6π	$3\pi/5$	$(1-\sqrt{5})/4$	$(1+\sqrt{5})/4$	$(5-\sqrt{5})/2$	$(5+\sqrt{5})/2$	5
4	8π	$4\pi/5$	$(-1-\sqrt{5})/4$	$(-1+\sqrt{5})/4$	$(3-\sqrt{5})/2$	$(3+\sqrt{5})/2$	3

Figure 2: Pairs of solutions and their sums

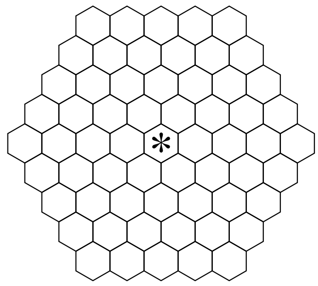
Problem 6. Let n be a positive integer and t be a non-zero real number. Let $a_1, a_2, \dots, a_{2n-1}$ be real numbers (not necessarily distinct). Prove that there exist distinct indices $i_1, i_2, \dots, i_n$ such that, for all $1 \le k, l \le n$, we have $a_{i_k} - a_{i_l} \neq t$.

Solution. Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, \dots, 2n-1\}$ and edge set $E = \{\{i, j\} : |a_i - a_j| = t\}$.

Note that G has no odd cycles. Indeed, if $j_1, \dots, j_{2k+1}$ is a cycle, then for all $\ell = 1, 3, 5, \dots, 2k-1$ the number a_{j_ℓ} differs from $a_{j_{\ell+2}}$ by $2t$ or 0. Hence a_{j_1} differs from $a_{j_{2k+1}}$ by an even multiple of t. Therefore there is no edge between j_1 and j_{2k+1} contradicting the assumption that $j_1, \dots, j_{2k+1}$ is a cycle.

Since G has no odd cycles, it is bipartite. Therefore V can be split into two disjoint sets V_1, V_2 such that there is no edge between any two vertices of V_1 and there are no edges between any two vertices in V_2. Since V has $2n-1$ elements, one of the sets V_1, V_2 has at least n elements. Without loss of generality assume that V_1 has at least n elements. Then for $k = 1, 2, \dots, n$ simply define i_k to be the k-th least element of V_1. □

Problem 7. Let $n > 2$ be an integer. Anna, Edda and Magni play a game on a hexagonal board tiled with regular hexagons, with n tiles on each side. The figure shows a board with 5 tiles on each side. The central tile is marked.



The game begins with a stone on a tile in one corner of the board. Edda and Magni are on the same team, playing against Anna, and they win if the stone is on the central tile at the end of any player's turn. Anna, Edda and Magni take turns moving the stone: Anna begins, then Edda and then Magni, and so on.

The rules for each player's turn are:

- Anna has to move the stone to an adjacent tile, in any direction.
- Edda has to move the stone straight by two tiles in any of the 6 possible directions.
- Magni has a choice of passing his turn, or moving the stone straight by three tiles in any of the 6 possible directions.

Find all n for which Edda and Magni have a winning strategy.

Solution. We colour the board in three colours in such a way that no neighbouring tiles are of the same colour. We can give each hexagon a coordinate using $\overrightarrow{e_1} = (1, 0)$ and $\overrightarrow{e_2} = (\cos(120°, \sin(120°)) = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ as basis. Let the central tile be the origin. Then each hexagon has center at $a \cdot \overrightarrow{e_1} + b \cdot \overrightarrow{e_2}$, $(a, b) \in \mathbb{Z}^2$. The tuple (a, b) is the coordinate for a given hexagon its neighbours are $(a + 1, b)$, $(a + 1, b + 1)$, $(a, b + 1)$, $(a - 1, b)$, $(a - 1, b - 1)$ and $(a, b - 1)$. We colour the hexagon with coordinates (a, b) with colour number $(a + b) \pmod 3$. It is clear that neighbouring hexagons do not share a colour. (In fact this is the only three colouring of a hexagonal tiling). See figure 3.

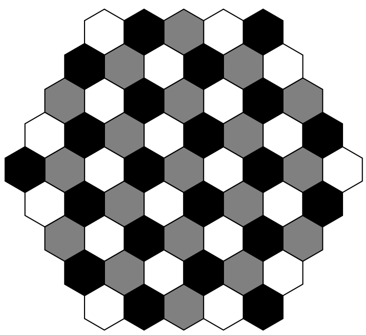


Figure 3: Three colouring of the hexagonal tiling for 5 hexagons on each side

We see that if $n \equiv 1 \pmod 3$, the stone begins in a tile in the same colour as the central tile, let that colour be grey. By regarding a few cases, we see that whatever Anna does, Edda and Magni can end their turns by getting the stone to a prescribed grey tile of the closest grey tiles. Therefore they can get the stone to the central tile.

If $n \not\equiv 1 \pmod 3$, the stone does not begin on the same grey colour as the central tile. Say the stone begins on a white tile, and say the third colour is black. Anna can always move the stone to a grey tile that is not on the same horizontal/diagonal line as the central tile. Then Anna moves the stone to a white or black tile. After Magni moves the stone is still again on a white/black tile. Anna can continue this indefinitely, with the stone never reaching the central tile.

$\square$

Problem 8. We are given a collection of 2^{2^k} coins, where k is a non-negative integer. Exactly one coin is fake. We have an unlimited number of service dogs. One dog is sick but we do not know which one. A test consists of three steps: select some coins from the collection of all coins; choose a service dog; the dog smells all of the selected coins at once. A healthy dog will bark if and only if the fake coin is amongst them. Whether the sick dog will bark or not is random.

Devise a strategy to find the fake coin, using at most $2^k + k + 2$ tests, and prove that it works.

Solution. Number the coins by 2^k-digit binary numbers from $\overbrace{00\ldots0}^{\text{length } 2^k}$ to $\overbrace{11\ldots1}^{\text{length } 2^k}$. Let A_i be the set of coins which have 0 in i-th position of the binary number. The first 2^k tests we perform with the help of 2^k different dogs. In the i-th test we determine whether the set A_i contains the fake coin. With out loss of generality we may assume that the dogs determined that all the digits in the number of the fake coin are 0's. Due to the possible presence of the sick dog in these tests, it means in fact that the binary number of the fake coin contains at most one 1.

In the next test we let a new dog determine whether the coin $\overbrace{00\ldots0}^{\text{length } 2^k}$ is genuine. If the new dog barks then the coin is really fake, for otherwise two dogs had given us a false answer. If the new dog does not bark we find a dog we have not used before to test the suspected coin.

(i) If the last two dogs disagree one of them must be sick and hence the first k dogs must be healthy. In this case the coin $\overbrace{00\ldots0}^{\text{length } 2^k}$ is the fake one.

(ii) If the last two dogs agree (by not barking) it follows that both of them are healthy. The reason is that if one of the last two dogs was sick and did not bark, it would mean that the first k dogs were healthy, implying that the coin $\overbrace{00\ldots0}^{\text{length } 2^k}$ is fake, but then the other of the last two dogs is healthy and did not bark at the fake coin, a contradiction.

Therefore one of the first 2^k dogs gave a wrong verdict. In this case we have 2^k possible candidates for the fake coin. We can find the fake coin using the last dog and k tests using binary search.

It follows that no more than $2^k + k + 2$ tests are needed. □

Problem 9. We are given 2021 points on a plane, no three of which are collinear. Among any 5 of these points, at least 4 lie on the same circle. Is it necessarily true that at least 2020 of the points lie on the same circle?

Solution. The answer is positive.

Let us first prove a lemma that if 4 points A, B, C, D all lie on circle Γ and some two points X, Y do not lie on Γ, then these 6 points are pairs of intersections of three circles, circle Γ and two other circles. Indeed, according to the problem statement there are 4 points among A, B, C, X, Y which are concyclic. These 4 points must include points X and Y because if one of them is not, then the other one must lie on Γ. Without loss of generality, take A, B, X, Y to lie on the same circle. Similarly for points A, C, D, X, Y there must be 4 points which are concyclic. Analogously, they must include points X and Y. Point A cannot be one of them because two circles cannot have more than two common points. Therefore, points C, D, X, Y are concyclic which proves the lemma.

Let us first solve the problem for the case for which there exist 5 points which lie on one circle Γ. Label these points A, B, C, D, E. Let us assume that there exists two points which do not lie on Γ, label them X and Y. According to the previously proven lemma, points A, B, C, D, X, Y must be the pairwise intersections of 3 circles. Without loss of generality, let the intersections of Γ with one of the other circles be A and B and with the other circle C and D. Similarly, A, B, C, E, X, Y must be the pairwise intersections of three circles one of which is Γ. This is not possible as none of the points A, B, C lies on the circumcircle of triangle EXY. This contradiction shows that at most 1 point can lie outside circle Γ, i.e. at least 2020 points lie on circle Γ.

It remains to look at the case for which no 5 points lie on the same circle. Let A, B, C, D, E be arbitrary 5 points. Without loss of generality, let A, B, C, D be concyclic and E a point not on this circle. According to the lemma, for every other point F and points A, B, C, D, E, the 6 points are the intersections of circle Γ and some two other circles. But in total, there are 3 such points because one of the two circles must go through E and some 2 points out of A, B, C, D, while the other circle must go through point E and the other two points out of A, B, C, D. There are only three partitions of A, B, C, D into two sets. There is a contradiction, as there are $2021 > 5 + 3$ points in total. □

Problem 10. John has a string of paper where n real numbers $a_i \in [0, 1]$, for all $i \in \{1, \dots, n\}$, are written in a row. Show that for any given $k < n$, he can cut the string of paper into k non-empty pieces, between adjacent numbers, in such a way that the sum of the numbers on each piece does not differ from any other sum by more than 1.

Solution 1. Denote the sums on each piece by

$$\begin{aligned} S_1 &= a_1 + a_2 + \ldots + a_{m_1}, \\ S_2 &= a_{m_1+1} + a_{m_1+2} + \ldots + a_{m_2}, \\ &\ldots \\ S_k &= a_{m_{k-1}+1} + \ldots + a_{m_k}. \end{aligned}$$

By abuse of notation S_i will both denote the set of numbers enclosed by cuts and its sum, the meaning of which must be determined by the context.

We will start the following algorithm. During this algorithm we will move some elements to the neighbouring piece and construct new sequence of pieces $S^\star = (S_1^\star, S_2^\star, \ldots, S_k^\star)$. Empty pieces may appear, but we will consider that case in the end.

(1) Find $p \leq k$ such that S_p is the piece with the maximum sum of elements.

(2) If $S_p \leq \min(S_1, \ldots, S_k) + 1$ we are done.

(3) If $S_p > \min(S_1, \ldots, S_k) + 1$, let S_q be the pieces with minimum sum of elements nearest to S_p (ties broken arbitrarily) and let S_h be the next pieces to S_q between S_p and S_q (it is non empty by the choice of S_q). Then either $p < q$ and then $h = q - 1$ and we define $S^\star$ by moving the last element from $S_h = S_{q-1}$ to S_q, or $q < p$, and then $h = q + 1$ and $S^\star$ is obtained by moving the first element of $S_h = S_{q+1}$ to S_q. If $p = h$ then set $S = S^\star$ and go to step (1). If $p \neq h$ then set $S = S^\star$ and proceed to step (2).

Note that in step (3) each number $S_i^\star$ is at most S_p and no new piece with sum S_p is created. Indeed, $S_h^\star < S_h \leq S_p$, and for some j we have $S_q^\star = S_q + a_j < S_p$ since $a_j \in [0, 1]$ and $S_p > \min(S_1, \ldots, S_k) + 1$. It is clear also that $\max(S_1, \ldots, S_k)$ does not increase during the algorithm.

Note also that in step (3) the pieces S_h may become empty. Then, in the next iteration of the algorithm, $q = h$ will be chosen since $\min(S_1, \ldots, S_k) = S_h = 0$ and in step (3) $S_h^\star$ will become non empty (but one of its neighbours may become empty, etc.).

Claim: In the algorithm above, Step (3) is repeated at most kn times with S_p being the same maximal pieces in $S^\star$ and in S.

Proof. Let s_i be the number of elements in i-th pieces. Then the number

$$\sum_{i=1}^{k} |i - p| s_i$$

takes positive integral values and is always less than kn. It is clear that this number decreases during the algorithm.

Thus after at most kn iteration of (3), the algorithm decreases the value of S_p and so goes to (1). Consequently it decreases either the number of pieces with maximal sums or $\max(S_1, \dots, S_k)$. As there are only finitely many ways to split the sum onto pieces, the algorithm eventually terminates at (2).

When the algorithm teminates, we are left with a sequence $S = (S_1, S_2, \dots, S_k)$ where the piece S_p with maximum sum is such that $S_p \leq \min(S_1, \dots, S_k) + 1$. If there is an empty piece S_h in the sequence, then the sum of any piece is in $[0, 1]$. We can for any empty piece create a cut in any place between numbers where there was not previously a cut, and discard the empty pieces. This is possible since the cuts are $k \leq n - 1$, and $n - 1$ is the number of places in between numbers. This operation will only possibly decrease the maximum sum, and still all sums will be in $[0, 1]$, so all conditions are satisfied. $\square$

Solution 2. This problem can be solved by finding a certain graph having a directed path of length k. For real x let (V_x, E_x) be a directed graph having vertices $V_x = \{0, 1, \dots, n\}$. If $i, j \in V_x$ we have a directed edge $(i, j) \in V_x$ iff $i \leq j$ and $\sum_{l=i+1}^{j} \in [x, x + 1]$.

Suppose that for some $x \in \mathbb{R}$ there exist such a graph (V_x, E_x) such that there exist a path of length k from vertex 0 to vertex n. Let $0 = v_0, v_1, \dots, v_k = n$ be an path of length k. If we cut the paper string between numbers v_i and $v_i + 1$ for $i \in \{1, \dots, k - 1\}$ we have that the sum of the i-th part is $a_{v_{i-1}+1} + \dots + a_{v_i} \in [x, x + 1]$. In particular the sum of the number of each piece does not differ from the sum of any other piece by more than 1.

It is therefore evident that the statement of the problem is equivalent to the existence of a real x such that there exists a path of length k form vertex 0 to vertex n in graph (E_x, V_x).

For a real x let s_x be the least vertex in (V_x, E_x) such that there exists a path of length k from 0 to s_x and let t_x be the greatest, provided that such a path exists. It is not difficult to prove (by induction on k) that for each vertex $v \in E_x$ such that $s_x \leq v \leq t_x$ there exists a path of length k for 0 to v.

Start with $x = 0$. As x increases both s_x and t_x increase. As there are only finitely many different sums formed by taking a subset of the a_i-s it follows that the graph (V_x, E_x) changes at discrete values of x. It is not hard to see that if the graph makes one change between reals $x_1 < x_2$ and $s_{x_1}, s_{x_2}, t_{x_1}, t_{x_2}$ exist then $s_{x_1} \leq s_{x_2} \leq t_{x_1} \leq t_{x_2}$.

For $s = \sum_{i=1}^{n} a_i + 1$ it is clear that there (V_s, E_s) has no edges. It is not hard to see that as x increases we will eventually find a value for x such that (V_x, E_x) has a path of length k for 0 to n as desired. $\square$

Problem 11. A point P lies inside a triangle ABC. The points K and L are the projections of P onto AB and AC, respectively. The point M lies on the line BC so that $KM = LM$, and the point P' is symmetric to P with respect to M. Prove that $\angle BAP = \angle P'AC$.

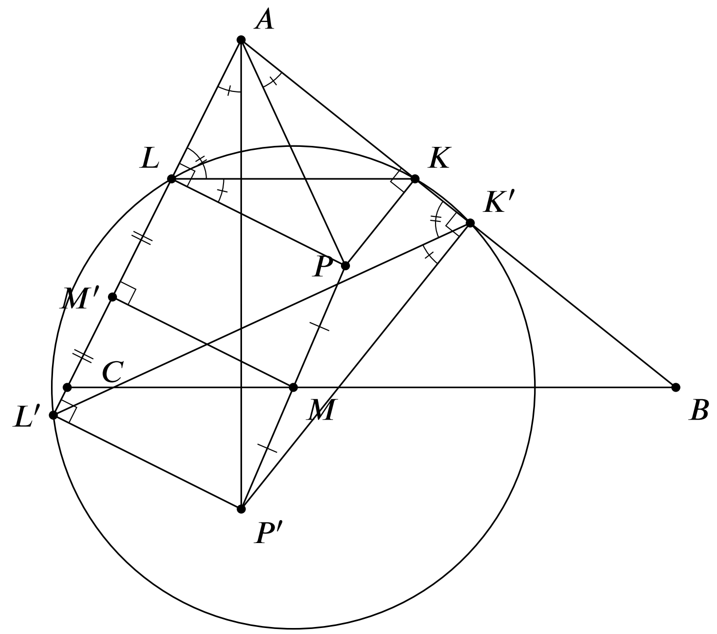


Figure 4

Solution. For points X, Y, Z, $X \neq Y$ and $Z \neq Y$ let rot XYZ denote the rotation that takes rotates line XY to line ZY modulo half turns. We consider two rotations equivalent one of them is a composition of some translation and the other rotation. It is clear that this is indeed an equivalence relation (as the Euclidean plane is Desarguean).

Let K' and L' be the projections of P' onto AB and AC respectively, as in figure 4. Let ℓ be the perpendicular line to line AB passing through M. From symmetries it follows that L' is the refection of L over ℓ. In particular segments ML and ML' are congruent. Similarly segments MK and MK' are congruent. It follows that M is a center of circle passing through L, K, L' and K'.

As line PL is perpendicular to line AC and line PK is perpendicular to line AB it follows that quadrilateral $AKPL$ is cyclic. Similarly quadrilateral $AK'PL'$ is also cyclic.

From the theorem on inscribed angles in cyclic quadrilaterals it follows that

$$\begin{gathered}\text{rot } BAP \equiv \text{rot } KAP \equiv \text{rot } KLP,\\ \text{rot } P'AC \equiv \text{rot } P'AL \equiv \text{rot } P'K'L \quad \text{and}\\ \text{rot } KLL' \equiv \text{rot } KK'L'\end{gathered}$$

As $\angle PLL'$ and $\angle KK'P'$ are right it follows that rot $PLL' \equiv$ rot $KK'P'$ modulo half turns. Now

$$\text{rot } KLL' \equiv \text{rot } KLP + \text{rot } PLL' \quad \text{and} \quad \text{rot } KK'L' \equiv \text{rot } P'K'L + \text{rot } KK'P$$

and rot $KLL' \equiv$ rot $KK'L'$ so we decuce that rot $KLP \equiv$ rot $P'K'L$.

Putting everything together gives

$$\text{rot } BAP \equiv \text{rot } P'AL'$$

which gives the desired result. $\square$

Remark. This method can be applied to prove the existence of isogonal conjugates in triangles.

Problem 12. Let I be the incentre of a triangle ABC. Let F and G be the projections of A onto the lines BI and CI, respectively. Rays AF and AG intersect the circumcircles of the triangles CFI and BGI for the second time at points K and L, respectively. Prove that the line AI bisects the segment KL.

Solution. Since $\angle IFK = 90°$, then IK is the diameter of the circumcircle of CFI, hence also $\angle ICK = 90°$. Similarly is IL the diameter of the circumcircle of BGI and $\angle IBL = 90°$. Therefore are the lines CK and GL parallel, also BL and FK are parallel.

Let the lines CK and BL intersect at D, as seen in figure 5. From the above we get that $DKAL$ is a parallelogram. Note that D is the excenter with respect to the vertex A of the triangle ABC, since the lines BL and CK are perpendicular to the corresponding internal angle bisectors. The excenter lies on the internal angle bisector AI, hence AI bisects the diagonal KL.

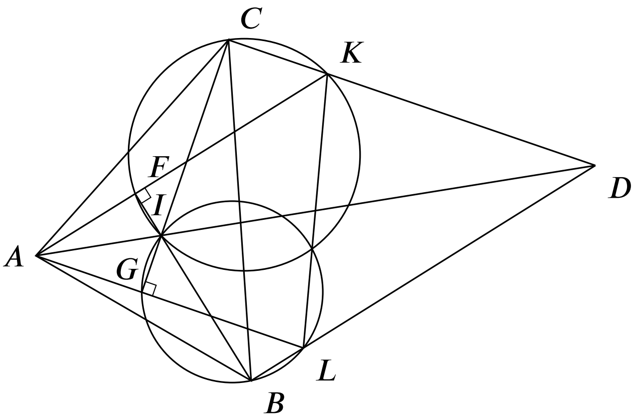


Figure 5

$\square$

Problem 13. Let D be the foot of the A-altitude of an acute triangle ABC. The internal bisector of the angle DAC intersects BC at K. Let L be the projection of K onto AC. Let M be the intersection point of BL and AD. Let P be the intersection point of MC and DL. Prove that $PK \perp AB$.

Solution 1. Let X be a point on BC such that $LX \perp AB$, as seen in figure 6. It is enough to prove that

$$\frac{DP}{PL} = \frac{DK}{KX}$$

because then $PK \parallel LX$ and $LX \perp AB$.

Applying Menelaos for triangle BDL and transversal MPC we get

$$\frac{DP}{PL} \cdot \frac{LM}{MB} \cdot \frac{BC}{CD} = 1,$$

and Menelaus for triangle BLC and transversal AMD gives

$$\frac{BM}{ML} \cdot \frac{LA}{AC} \cdot \frac{CD}{DB} = 1.$$

Multiplying these two equalities yields

$$\frac{DP \cdot BC \cdot AL}{PL \cdot BD \cdot AC} = 1.$$

Note, however, that $AL = AD = AC \sin\gamma$, $BD = AB\cos\beta$, and, by the sine rule, $\frac{AB}{BC} = \frac{\sin\gamma}{\sin\alpha}$, where $\alpha = \angle BAC, \beta = \angle CBA$ and $\gamma = \angle ACB$. Therefore

$$\frac{DP}{PL} = \frac{BD \cdot AC}{BC \cdot AL} = \frac{AB\cos\beta \cdot AC}{BC \cdot AC\sin\gamma} = \frac{\sin\gamma\cos\beta}{\sin\alpha\sin\gamma} = \frac{\cos\beta}{\sin\alpha}.$$

On the other hand, since $DK = KL$, $\angle KLX = \pi - \alpha$, and $\angle LXK = \frac{\pi}{2} - \beta$, we have by the sine rule

$$\frac{DK}{KX} = \frac{LK}{KX} = \frac{\sin(\frac{\pi}{2} - \beta)}{\sin(\pi - \alpha)} = \frac{\cos\beta}{\sin\alpha}.$$

Therefore

$$\frac{DP}{PL} = \frac{\cos\beta}{\sin\alpha} = \frac{DK}{KX}$$

which finishes the proof. $\square$

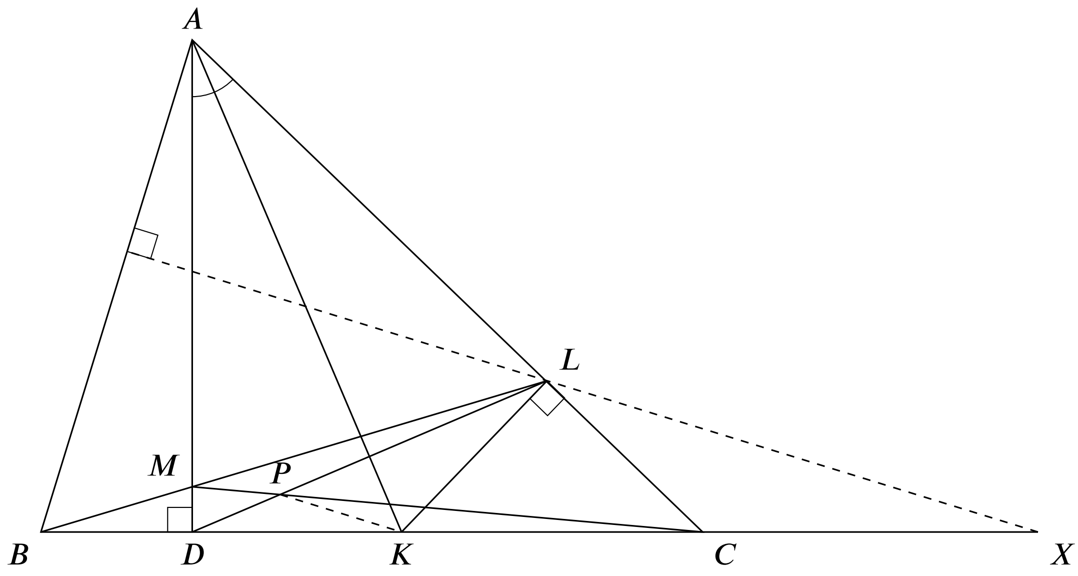


Figure 6

Solution 2. Let ω be the circle with center K an radius KD, as in figure 7. Then ω is tangent to AD and AL. Let BC intersect ω at D and Q. Let BM intersect ω at L and R. Let QP intersect BL at S.

Cross-ratio chasing gives, through the projections $BL \to D$-pencil $\to \omega \to L$-pencil $\to BC \to P$-pencil $\to BL$,

$$\begin{aligned}(L, R; M, B) &= (DL, DR; DM, DB) = (L, R; D, Q) = (LC, LB; LD, LQ)\\ &= (C, B; D, Q) = (PC, PB; PD, PQ) = (M, B; L, S) = (L, S; M, B),\end{aligned}$$

therefore $R = S$.

It is clear now that P lies on the polar lines of both A and B with respect to ω, therefore AB is the polar line of P. This implies that $PK \perp AB$.

□

Problem 14. Let ABC be a triangle with circumcircle Γ and circumcentre O. Denote by M the midpoint of BC. The point D is the reflection of A over BC, and the point E is the intersection of Γ and the ray MD. Let S be the circumcentre of the triangle ADE. Prove that the points A, E, M, O, and S lie on the same circle.

Solution. First we prove that A, M, E, S are concyclic. Note that BC is the perpendicular bisector of AD, so S lies on BC. Let X be the intersection of AD and BC as in figure 8. Then, using directed

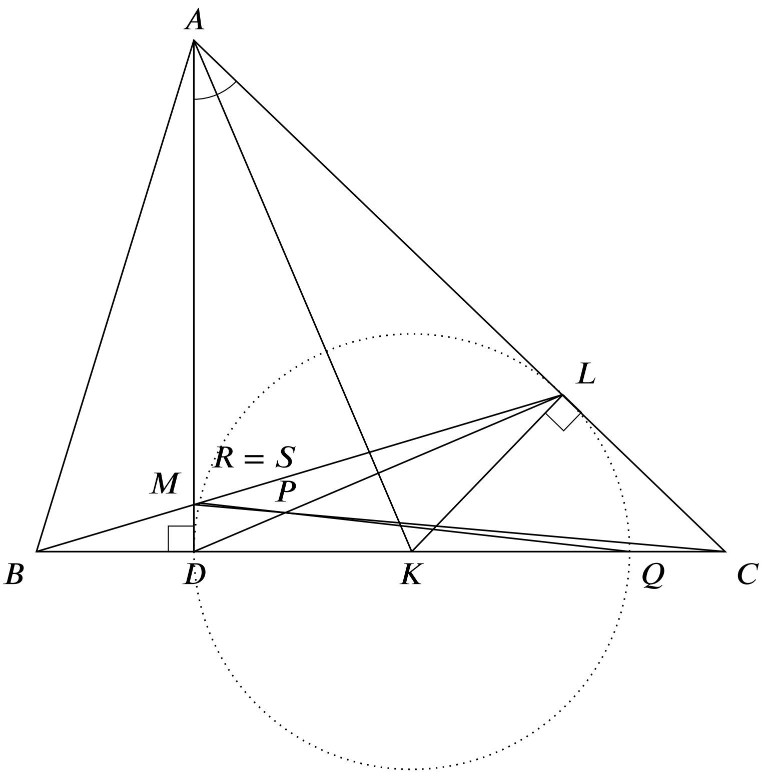


Figure 7

angles,

$$\begin{aligned}\angle EMS &= \angle DMX \\ &= 90° - \angle XDM \\ &= 90° - \angle ADE \\ &= \angle SEA \\ &= \angle EAS,\end{aligned}$$

so $AMES$ is cyclic as claimed.

Now we prove that A, M, E, O are concyclic. Let F be the reflection of D over M. Then F lies on the same side of BC as A and satisfies $FCB \cong DBC \cong ABC$, so F must be the point such that $AFCB$ is an isosceles trapezoid. In particular, F lies on Γ. Consequently,

$$\angle OAE = 90° - \angle EFA = 90° - \angle EMB = \angle OMB + \angle BME = \angle OME,$$

so $AMEO$ is also cyclic as claimed.

Thus A, E, M, O, S are concyclic, as desired.

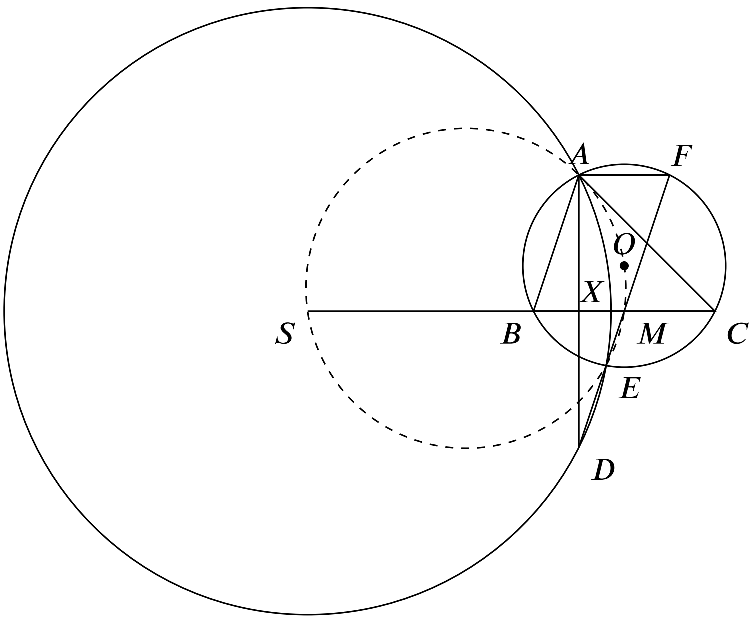


Figure 8

□

Problem 15. For which positive integers $n \geq 4$ does there exist a convex n-gon with side lengths $1, 2, \dots, n$ (in some order) and with all of its sides tangent to the same circle?

Solution. It exists if $n = 4k$ or $n = 4k + 1$ where k is a positive integer.

Let us consider n-gon $P_1 P_2 \dots P_n$. Tangent points of the inscribed circle divide each of its sides in two segments. Lengths of these segments that has a common vertex P_i are equal. Denote the length of tangent segments that originate at point P_i by A_i. It means that side lengths of the n-gon can be expressed as $P_i P_{i+1} = A_i + A_{i+1}$ for all $i = 1, 2, \dots, n$ where we consider points cyclically ($P_{n+1} = P_1$ and $A_{n+1} = A_1$).

We can show that the converse is true as well. That is, if we can find n positive real numbers A_i, $i = 1, 2, \dots, n$ such that the sequence $(A_1 + A_2, A_2 + A_3, \dots, A_n + A_1)$ is a permutation of $(1, 2, \dots, n)$ then there is a circumscribed polygon $P_1 P_2 \dots P_n$ with side lengths $1, 2, \dots, n$.

To show this we start with a circle of arbitrary radius R and construct points $P_1, P_2, \dots, P_n$ outside this circle so that the length of the tangent segments from P_i to the circle are of length A_i and the "right" tangent segment from P_i touches the circle at the same point as the "left" tangent segment from P_{i-1}.

Now we almost have the n-gon except that possibly the "right" tangent point of P_1 does not match the "left" touching point of P_n. This can be easily fixed by adjusting the radius R of the circle, using continuity.

Now we solve the problem by considering 4 cases:

(i) First let's consider the case when $n = 4k$. In this case such circumscribed n-gon exists. The $4k$ segments A_i can be of lengths

$$A_1 = \frac{1}{2}, A_2 = \frac{1}{2}, A_3 = \frac{3}{2}, A_4 = \frac{3}{2}, \dots, A_{2k-1} = \frac{2k-1}{2}, A_{2k} = \frac{2k-1}{2},$$
$$A_{2k+1} = \frac{2k+1}{2}, A_{2k+2} = \frac{6k-1}{2}, A_{2k+3} = \frac{2k-1}{2}, A_{2k+4} = \frac{6k-3}{2}, \dots,$$
$$A_{4k-1} = \frac{3}{2}, A_{4k} = \frac{4k+1}{2}.$$

One can see that the values of the sums of the consecutive elements $A_1 + A_2, A_2 + A_3, \dots, A_{4k-1} + A_{4k}, A_{4k} + A_{4k+1}$ are exactly $1, 2, \dots, 2k, 4k, 4k-1, \dots, 2k+1$, respectively.

(ii) In the case $n = 4k + 1$ the construction is similar, we can choose $4k + 1$ segments of length

$$A_1 = \frac{1}{2}, \quad A_2 = \frac{1}{2}, \quad A_3 = \frac{5}{2}, \quad A_4 = \frac{5}{2}, \dots,$$
$$A_{2k+1} = \frac{4k+1}{2}, \quad A_{2k+2} = \frac{4k+1}{2}, \quad A_{2k+3} = \frac{4k-1}{2},$$
$$A_{2k+4} = \frac{4k-3}{2}, \quad A_{2k+5} = \frac{4k-5}{2}, \quad \dots, A_{4k+1} = \frac{3}{2}.$$

In this case the values of the sums of consecutive elements $A_1 + A_2$, $A_2 + A_3$, $\dots$, $A_{4k-1} + A_{4k}$, $A_{4k} + A_{4k+1}$ are $1, 3, 5, \dots, 4k+1, 4k, 4k-2, \dots, 2$, respectively.

(iii) In case when $n = 4k + 2$ such a polygon does not exist. To prove this we note that in case if the number of the sides of the circumscribed polygon is even then the sum of the odd numbered sides is equal to the sum of the even numbered sides. It is evident as two segments of equal length that originate from the same vertex contribute to different sums. But the total sum of the side lengths is an odd number what means that it is impossible to split the sides on two parts with equal sum of lengths.

(iv) In case $n = 4k + 3$ such a polygon also does not exist. In this case we can express A_1 as

$$\begin{aligned} A_1 &= (A_1 + A_2 + \ldots + A_n) - (A_2 + A_3) - (A_4 + A_5) - \ldots - (A_{4k+2} + A_{4k+3}) = \\ &= \frac{P_1P_2 + P_2P_3 + \cdots + P_{4k+3}P_1}{2} - P_2P_3 - P_4P_5 - \ldots - P_{4k+2}P_{4k+3} \end{aligned}$$

As the sum of the length of the sides is an even number then we conclude that A_1 is a positive integer. The same is true for all $A_2, A_3, \ldots$ as well. But now we have a contradiction as the side of length 1 cannot be split in two parts, each of which has positive integer length.

$\square$

Problem 16. Show that no non-zero integers a, b, x, y satisfy

$$\begin{cases} ax - by = 16, \\ ay + bx = 1. \end{cases}$$

Solution. If we use the Diophantus sum of squares equality

$$(ax - by)^2 + (ay + bx)^2 = (a^2 + b^2)(x^2 + y^2)$$

then we can see that for a system

$$\begin{cases} ax - by = s \\ ay + bx = t \end{cases}$$

to have a solution in positive integers the number $s^2 + t^2$ must be a composite number.

The number corresponding to the equation, $16^2 + 1^2 = 257$, is a prime number. This shows that no solution can exist in non-zero integers, as it would give a factorisation of the prime with each factor > 1. $\square$

Remark. Note that $(a + bi)(x + yi) = (ax - by) + (ay + bx)i$. Finding a solution to the system of equations is therefore equivalent to finding a factorization of $s + ti$ in Gaussian integers $\mathbb{Z}[i]$ with non-negative real and imaginary component. It is known that the Gaussian integers form a Euclidean domain and hence a unique factorization domain. The form for primes in $\mathbb{Z}[i]$ has been thoroughly studied.

Problem 17. Distinct positive integers a, b, c, d satisfy

$$\begin{cases} a \mid b^2 + c^2 + d^2, \\ b \mid a^2 + c^2 + d^2, \\ c \mid a^2 + b^2 + d^2, \\ d \mid a^2 + b^2 + c^2, \end{cases}$$

and none of them is larger than the product of the three others. What is the largest possible number of primes among them?

Solution. At first we note that the given condition is equivalent to $a, b, c, d | a^2 + b^2 + c^2 + d^2$.

It is possible that three of the given numbers are primes, for example for $a = 2$, $b = 3$, $c = 13$ and $d = 26$. In this case $2^2 + 3^2 + 13^2 + 26^2 = 13 \cdot 66$ which is divisible by all four given numbers. Furthermore we will show that it is impossible that all four of them are primes.

Let us assume that a, b, c and d are primes. As the sum $a^2 + b^2 + c^2 + d^2$ is divisible by each of them then it is divisible also by their product $abcd$. If one of the primes is equal to 2, then we obtain a contradiction: the sum of four squares is odd, but its divisor $abcd$ is even. Therefore all four primes are odd, and $a^2 + b^2 + c^2 + d^2 = 0 \pmod 4$. Hence $a^2 + b^2 + c^2 + d^2$ is divisible by $4abcd$ which leads to a contradiction as it is easy to see that $a^2 + b^2 + c^2 + d^2 < 4abcd$. Indeed, this is equivalent to

$$\frac{a}{bcd} + \frac{b}{acd} + \frac{c}{abd} + \frac{d}{abc} < 4$$

which is true as none of the numbers exceed the product of three three other and equality can hold only for the largest of the four. □

Problem 18. Find all integer triples (a, b, c) satisfying the equation

$$5a^2 + 9b^2 = 13c^2.$$

Solution. Observe that $(a, b, c) = (0, 0, 0)$ is a solution. Assume that the equation has a solution $(a_0, b_0, c_0) \neq (0, 0, 0)$. Let $d = \gcd(a_0, b_0, c_0) > 0$. Let $(a, b, c) = (a_0/d, b_0/d, c_0/d)$. Then $\gcd(a, b, c) = 1$. From $5{a_0}^2 + 9{b_0}^2 = 13{c_0}^2$ it follows that:

$$5a^2 + 9b^2 = 5\left(\frac{a_0}{d}\right)^2 + 9\left(\frac{b_0}{d}\right)^2 = \frac{5{a_0}^2 + 9{b_0}^2}{d^2} = \frac{13{c_0}^2}{d^2} = 13\left(\frac{c_0}{d}\right)^2 = 13c^2$$

hence (a, b, c) is also a solution.

As $(a_0, b_0, c_0) \neq (0, 0, 0)$ it follows that $(a, b, c) \neq (0, 0, 0)$. Consider the equation modulo 5. It follows that $4b^2 \equiv 5a^2 + 9b^2 = 13c^2 \equiv 3c^2 \pmod 5$, that is $4b^2 \equiv 3c^2 \pmod 5$. Multiplying by 4 gives:

$$b^2 \equiv 16b^2 = 4 \cdot 4b^2 \equiv 4 \cdot 3c^2 = 12c^2 \equiv 2 \cdot c^2 \pmod 5$$

If $5|c$ then $2c^2 \equiv 2 \cdot 0^2 = 0 \pmod 5$ and therefore $a^2 \equiv 0 \pmod 5$, that is $5|b^2$. As 5 is prime it follows that $5|b$. Hence 5 divides b and c. It follows that $5^2|13c^2 - 9b^2 = 5a^2$. Consequently 5 divides a^2. As 5 is prime, $5|a$. This means that 5 divides a, b and c contradicting the fact that $\gcd(a, b, c) = 1$. We conclude that $5|c$ does not hold.

As $5|c$ does not hold and 5 is a prime it follows that c and 5 are relative prime. Therefore there exists $x \in \mathbb{Z}$ such that $c \cdot x \equiv 1 \pmod 5$. Multiplying by x^2 gives:

$$(b \cdot x)^2 = b^2 \cdot x^2 \equiv 2 \cdot c^2 \cdot x^2 = 2 \cdot (c \cdot x)^2 \equiv 2 \cdot 1^2 = 2 \pmod 5$$

That is $y^2 \equiv 2 \pmod 5$ where $y = b \cdot x$. As $y^2 \equiv 2 \pmod 5$ it follows that y and 5 are relative prime. By Fermat's little theorem it follows that $y^4 \equiv 1 \pmod 5$. Hence:

$$1 \equiv y^4 = (y^2)^2 \equiv 2^2 = 4 \pmod 5$$

but $1 \not\equiv 4 \pmod 5$ so we have a contradiction. We conclude that the equation $5a^2 + 9b^2 = 13c^2$ has no solution besides the solution $(a, b, c) = (0, 0, 0)$. $\square$

Problem 19. Find all polynomials p with integer coefficients such that the number $p(a) - p(b)$ is divisible by $a + b$ for all integers a, b, provided that $a + b \neq 0$.

Solution. The polynomials we are looking for are those whose every odd-degree term has zero coefficient.

Let $P(x) = P_0(x) + P_1(x)$, where P_0 and P_1 are polynomials whose all non-zero terms have either even or odd degree, respectively.

Then we can write $P_0(x) = Q(x^2)$, where polynomial Q is obtained from polynomial P_0 by dividing degrees of all non-zero terms by 2. Now, for any integers a, b the number $P_0(a) - P_0(b) = Q(a^2) - Q(b^2)$

is divisible by $a^2 - b^2$, and hence also by $a + b$. Thus, if every odd-degree term of P has zero coefficient, then the condition of the problem is satisfied.

On the other hand, if polynomial P satisfies the condition of the problem, then also $P - P_0 = P_1$ must satisfy it. Note that for every real x, $P_1(-x) = -P_1(x)$, i.e. P_1 is an odd function. By substituting b by $-b$ in the condition of the problem we obtain that $a - b | P_1(a) + P_1(b)$ holds for any distinct integers a and b. Since also $a - b | P_1(a) - P_1(b)$, then for any integers a, b we have $a - b | 2P_1(a)$. But for any a there exists such b that $|a - b| > 2P_1(a)$. From this we conclude that $P_1(a) = 0$ for any integer a. Altogether we have $P = P_0$, i.e. coefficients of all odd-degree terms are zero. □

Problem 20. Let $n \geq 2$ be an integer. Given numbers $a_1, a_2, \dots, a_n \in \{1, 2, 3, \dots, 2n\}$ such that $\operatorname{lcm}(a_i, a_j) > 2n$ for all $1 \leq i < j \leq n$, prove that

$$a_1 a_2 \cdots a_n \mid (n+1)(n+2) \cdots (2n-1)(2n).$$

Solution. For every $i = 1, 2, \dots, n$ let $b_i = \max\{k \cdot a_i | k \in \mathbb{Z},\ k \cdot a_i \leq 2n\}$, that is b_i is the greatest multiple of a_i in $\{1, 2, \dots, 2n\}$. It is clear that $b_i \in \{n+1, n+2, \dots, 2n\}$ because if $b_i \leq n$ then $2b_i \leq 2n$ is a greater multiple of a_i.

If $1 \leq i < j \leq n$ then $a_i \nmid b_j$ because else $b_j \leq 2n$ would be a common multiple of a_i and a_j smaller than their least common multiple $\operatorname{lcm}(a_i, a_j) > 2n$. In particular $b_i \neq b_j$. It follows that the map $\{1, 2, \dots, n\} \to \{n+1, n+2, \dots, 2n\}$, $i \mapsto b_i$ is injective. As both sets $\{1, 2, \dots, n\}$ and $\{n+1, n+2, \dots, 2n\}$ have the same finite cardinality it follows that the map is also a surjection and hence a bijection. In particular $b_1 b_2 \cdots b_n = (n+1)(n+2) \cdots (2n)$ by associativity and commutativity.

As each $a_i | b_i$ it follows that

$$a_1 a_2 \cdots a_n | b_1 b_2 \cdots b_n = (n+1)(n+2) \cdots (2n)$$

as desired. □