Mathematics Competition for the Seventh Graders of Helsinki 2012/1/18

Problems and Solutions

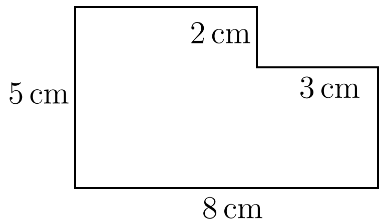
(1) Compute $6 \cdot 5 \cdot 4 - 5 \cdot 4 \cdot 3 + 4 \cdot 3 \cdot 2 - 3 \cdot 2 \cdot 1$.

a) 88 **b)** 66 **c)** 78 **d)** 76

Solution. A direct computation:

$$6 \cdot 5 \cdot 4 - 5 \cdot 4 \cdot 3 + 4 \cdot 3 \cdot 2 - 3 \cdot 2 \cdot 1 = 120 - 60 + 24 - 6 = 60 + 18 = 78.$$

(2) Compute the circumference of the following figure.



a) 18 cm **b)** 25 cm **c)** 26 cm **d)** 30 cm.

Solution. The circumference of the figure is the same as the circumference of a $8\,\text{cm} \times 5\,\text{cm}$-rectangle, i.e.

$$8\,\text{cm} + 8\,\text{cm} + 5\,\text{cm} + 5\,\text{cm} = 26\,\text{cm}.$$

(3) The base of an isosceles triangle is 5 and its area is 45. What is its height?

a) 4.5 **b)** 18 **c)** 9 **d)** 112.5

Solution. The area of a triangle is one half of the product of its base and its height, which in this case is 45. Therefore the product of the base and the height is $2 \cdot 45 = 90$. Now the height must be this number divided by the base, or $\frac{90}{5} = 18$.

(4) The sum of three consecutive integers is 42. What is the middle one?

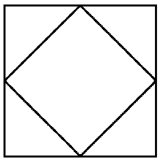
a) 13 **b)** 14 **c)** 15 **d)** 16.

Solution. If the numbers are $x - 1$, x and $x + 1$, then

$$42 = (x - 1) + x + (x + 1) = 3x.$$

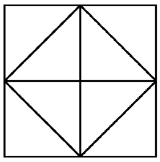
Therefore $x = \dfrac{42}{3} = 14$.

(5) The midpoints of neighboring sides of a $1\,\mathrm{m} \times 1\,\mathrm{m}$-square have been connected with line segments, and we have thereby obtained a smaller square inside the original one. What is the area of the smaller square?



a) $0.25\,\mathrm{m}^2$ **b)** $0.5\,\mathrm{m}^2$ **c)** $1\,\mathrm{m}^2$ **d)** $2\,\mathrm{m}^2$

Solution. Let us divide the figure into identical small triangles as follows:



Now the area of the larger square is eight small triangles whereas the area of the smaller square is four small triangles. Thus the area of the smaller square is one half of the area of the larger square, or $0.5\,\mathrm{m}^2$.

(6) In order to build a small forest cabin, we need one hundred logs, each of which must be five meters long. In the beginning, we have only logs which are twenty meters long each. What is the smallest number of times we have to saw through a log in order to obtain the five meter logs we need?

a) 50 **b)** 75 **c)** 99 **d)** 100

Solution. By cutting three times we obtain four five meter logs from one twenty meter log. Therefore we obtain $100 = 25 \cdot 4$ five meter logs by making $25 \cdot 3 = 75$ cuts.

(7) What is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$?

a) 120 **b)** 720 **c)** 5040 **d)** 40320

Solution. A direct calculation:

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 6 \cdot 20 \cdot 42 = 120 \cdot 42 = 5040.$$

(8) Jack is climbing a beanstalk. The beanstalk is $88\,\mathrm{m}$ long. Each time Jack has climbed five meters, an ogre shakes the beanstalk and Jack slides down one meter. When Jack has reached the top, the ogre can no more shake him down. How many meters does Jack climb altogether?

a) $88\,\mathrm{m}$ **b)** $100\,\mathrm{m}$ **c)** $109\,\mathrm{m}$ **d)** $110\,\mathrm{m}$

Solution. Until the very end Jack needs to climb 5 meters for each 4 meters he actually moves ahead. When he reaches the height of $84\,\mathrm{m}$, he has climbed

$\frac{5}{4} \cdot 84 = 5 \cdot 21 = 105$ meters. But now he reaches the top of the beanstalk simply by climbing four more meters, so that, all in all, he climbs $105 + 4 = 109$ meters.

(9) A concert is organized in the Hartwall Arena. The organizers estimate that if the price of a ticket is x euros, then the fans will buy $10000 + 400x - 10x^2$ tickets. The organizers have to choose between two prices: 30 euros per ticket and 40 euros per ticket. Which choice brings more people to the concert? Which choice earns more money for the organizers?

a) 30 euros brings more people and more money.
b) 30 euros brings more people and 40 euros brings more money.
c) 40 euros brings more people and 30 euros brings more money.
d) 40 euros brings more people and more money.

Solution. According to the estimates of the organizers, 30 euro tickets would bring

$$10000 + 400 \cdot 30 - 10 \cdot 30^2 = 10000 + 12000 - 9000 = 13000$$

people to the concert, who would then pay

$$13000 \cdot 30 = 390000 \text{ euros}$$

for their tickets.

The 40 euro option would bring

$$10000 + 400 \cdot 40 - 10 \cdot 40^2 = 10000 + 16000 - 16000 = 10000$$

people, who would pay

$$10000 \cdot 40 = 400000 \text{ euros.}$$

We conclude that the 30 euro choice would bring more people and the 40 euro choice would bring more money.

(10) Ville went into a five-storey building in order to sell chocolate eggs. The resident of the topmost floor bought half of the eggs and a half of an egg on top of that. The resident of the fourth floor bought half of the remaining eggs and a half of an egg on top of that. The resident of the third floor, the resident of the second floor and the resident of the first floor each did the same thing. After all this Ville noticed that he had sold all of his eggs. How many chocolate eggs did Ville have in the beginning?

a) 7 **b)** 15 **c)** 23 **d)** 31

Solution. Since the first floor resident bought half of the remaining eggs and only half on an egg remained, we conclude that Ville must have had precisely one egg left when he descended down to the first floor.

Similarly, after the second floor resident bought half of the remaining eggs but before he bought half of an egg, Ville must have had one and a half eggs. Therefore Ville must have had $2 \cdot \left(1 + \frac{1}{2}\right) = 3$ eggs when he arrived on the second floor.

In the same vein, Ville must have had $2 \cdot \left(3 + \frac{1}{2}\right) = 7$ eggs when he arrived on the third floor, $2 \cdot \left(7 + \frac{1}{2}\right) = 15$ eggs when he arrived on the fourth floor, and in the very beginning he must have had $2 \cdot \left(15 + \frac{1}{2}\right) = 31$ eggs.

(11) Let $X = 1 + 2 + 3 + 4 + \cdots + 70$. How large is X?

a) 1001 **b)** 2485 **c)** 3110 **d)** 4953

Solution. Let us consider the number $2X$: it is the sum of the numbers

$$\begin{array}{rrrrcrr} 1 & 2 & 3 & 4 & \cdots & 69 & 70 \\ 70 & 69 & 68 & 67 & \cdots & 2 & 1 \end{array}$$

But this sum must clearly be $70 \cdot 71 = 4970$. Hence

$$X = \frac{4970}{2} = 2485.$$

(12) Let us consider the number $N = 11 \cdot 11 \cdot \ldots \cdot 11$, where there are 2012 multiplicands. What are the last two digits of N?

a) 11 **b)** 21 **c)** 31 **d)** 41

Solution. The idea behind this question is that the last two digits of the product of two natural numbers only depend on the last digits of each of the multiplicands. Therefore it suffices that we consider what happens to the last two digits in the multiplications.

Let us begin:

$$\begin{cases} 11 \cdot 11 = 121, & 11 \cdot 51 = 561, \quad 11 \cdot 91 = 1001, \\ 11 \cdot 21 = 231, & 11 \cdot 61 = 671, \quad 11 \cdot 1 = 11. \\ 11 \cdot 31 = 341, & 11 \cdot 71 = 781, \\ 11 \cdot 41 = 451, & 11 \cdot 81 = 891, \end{cases}$$

Thus, in the sequence $11, 11 \cdot 11, 11 \cdot 11 \cdot 11, \ldots$ the pairs of last two digits form a self-repeating sequence:

$$11, \quad 21, \quad 31, \quad 41, \quad 51, \quad 61, \quad 71, \quad 81, \quad 91, \quad 01, \quad 11, \quad \ldots$$

In this sequence, the tenth, the twentieth, the thirtieth, and so on, terms are equal to 01, which means that the product $11 \cdot 11 \cdot \ldots \cdot 11$ with 2010 multiplicands ends with the digits 01. The next product must then end with 11, and the product with 2012 multiplicands has to end with the digits 21.

(13) What is $\dfrac{789}{999} - \dfrac{12}{99}$?

a) $0.668688\ldots$ **b)** $0.666\ldots$ **c)** $0.668577\ldots$ **d)** $0.668668\ldots$

Solution. Once we have somehow concluded that

$$\frac{789}{999} = 0{,}789789789\ldots \quad \text{and} \quad \frac{12}{99} = 0{,}12121212\ldots,$$

the rest is easy:

$$\frac{789}{999} - \frac{12}{99} = 0{,}789789789\ldots - 0{,}12121212\ldots = 0{,}668577\ldots$$

It is easy to justify the above values for the fractions $\frac{789}{999}$ and $\frac{12}{99}$: after all, we have

$$\begin{aligned} 999 \cdot 0{,}789789\ldots &= 1000 \cdot 0{,}789789\ldots - 0{,}789789\ldots \\ &= 789{,}789789\ldots - 0{,}789789 = 789, \end{aligned}$$

and

$$\begin{aligned} 99 \cdot 0{,}1212\ldots &= 100 \cdot 0{,}1212\ldots - 0{,}1212\ldots \\ &= 12{,}1212\ldots - 0{,}1212\ldots = 12. \end{aligned}$$

(14) Let $X = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \frac{1}{3125} + \frac{1}{15625}$. What can we say about X?

a) $0 < X \leqslant \frac{1}{4}$ **b)** $\frac{1}{4} < X \leqslant \frac{1}{2}$ **c)** $\frac{1}{2} < X \leqslant \frac{3}{4}$ **d)** $\frac{3}{4} < X \leqslant 1$

Solution. We observe that all the denominators are powers of the number five:

$$X = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} + \frac{1}{5^6}.$$

Now

$$\frac{1}{5} + \frac{X}{5} = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} + \frac{1}{5^6} + \frac{1}{5^7} = X + \frac{1}{5^7}.$$

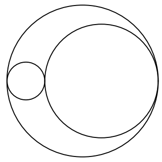
Hence $\frac{1+X}{5} > X$ ja $1 + X > 5X$, from which it follows that $1 > 4X$ and also that $X < \frac{1}{4}$.

Alternatively we could just expand all the terms in order to have equal denumerators:

$$X = \frac{3125 + 625 + 125 + 25 + 1}{15625} = \frac{3951}{15625}.$$

Since $4 \cdot 3951 = 14004 < 15625$, we must have $X < \frac{1}{4}$.

(15) A region of the shape of a circle has area equal to 80. As in the following picture, we remove from it two regions of the shape of a circle, the other having diagonal equal to one fourth and the other having diagonal equal to three fourths of the diagonal of the original circle.



What is the area of the remaining region?

a) 20 **b)** 30 **c)** 40 **d)** 50

Solution. When the scale is halved, areas are reduced to one quarter. When the scale is diminished to one fourth, areas must reduce to one sixteenth. Thus the area of the smaller circle is $\frac{80}{16} = 5$.

In the same vein, when the scale is tripled, areas grow ninefold, and the area of the second largest circle has to be $9 \cdot 5 = 45$.

The area of the remaining region must now be $80 - 5 - 45 = 30$.