Mathematics Competition for the Seventh Graders of Oulu 2012/1/18

Problems and Solutions

(1) The sum of three consecutive integers is 42. What is the middle one?

a) 13 **b)** 14 **c)** 15 **d)** 16.

Solution. If the numbers are $x-1$, x and $x+1$, then

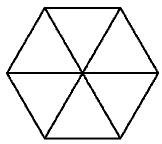
$$42 = (x-1) + x + (x+1) = 3x.$$

Therefore $x = \frac{42}{3} = 14$.

(2) The length of a side of a regular hexagon is 5. What is the length of its diagonal (from one vertex to the opposing vertex)?

a) 5 **b)** $5\sqrt{3}$ **c)** 10 **d)** $10\sqrt{3}$

Solution. We may divide the regular hexagon into identical equilater triangles as follows:



Now it is obvious that the diagonal of the regular hexagon must be twice as long as its sides are, which means that the length of the diagonal must be $2 \cdot 5 = 10$.

(3) Compute $9 \cdot 8 - 8 \cdot 7 + 7 \cdot 6 - 6 \cdot 5$.

a) 38 **b)** -28 **c)** -38 **d)** 28

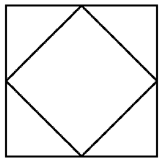
Solution. We may compute directly as follows:

$$9 \cdot 8 - 8 \cdot 7 + 7 \cdot 6 - 6 \cdot 5 = 72 - 56 + 42 - 30 = 16 + 12 = 28.$$

We could also benefit from the common factors of the terms:

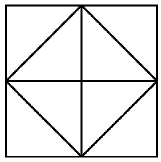
$$\begin{aligned} 9 \cdot 8 - 8 \cdot 7 + 7 \cdot 6 - 6 \cdot 5 &= (9-7) \cdot 8 + (7-5) \cdot 6 \\ &= 2 \cdot 8 + 2 \cdot 6 = 16 + 12 = 28. \end{aligned}$$

(4) The midpoints of neighboring sides of a $1\,\mathrm{m} \times 1\,\mathrm{m}$-square have been connected with line segments, and we have thereby obtained a smaller square inside the original one. What is the area of the smaller square?



a) $0.25\,\mathrm{m}^2$ **b)** $0.5\,\mathrm{m}^2$ **c)** $1\,\mathrm{m}^2$ **d)** $2\,\mathrm{m}^2$

Solution. Let us divide the figure into identical small triangles as follows:



Now the area of the larger square is eight small triangles whereas the area of the smaller square is four small triangles. Thus the area of the smaller square is one half of the area of the larger square, or $0.5\,\mathrm{m}^2$.

(5) What are the last two digits of the number $25 \cdot 25 \cdot \ldots \cdot 25$?

a) 25 **b)** 35 **c)** 45 **d)** 55

Solution. The key idea of the problem is in the observation that the last two digits of the product of two natural numbers only depend on the last two digits of each of the multiplicands. Since the product $25 \cdot 25 = 625$ ends with the digits 25, all products of arbitrary natural numbers ending with the digits 25 must also end with the digits 25. In particular, the product $25 \cdot 25 \cdot \ldots \cdot 25$ must end with the digits 25.

(6) In order to build a small forest cabin, we need one hundred logs, each of which must be five meters long. In the beginning, we only have logs which are twenty meters long each. What is the smallest number of times we have to saw through a log in order to obtain the five meter logs we need?

a) 50 **b)** 75 **c)** 99 **d)** 100

Solution. By cutting three times we obtain four five meter logs from one twenty meter log. Therefore we obtain $100 = 25 \cdot 4$ five meter logs by making $25 \cdot 3 = 75$ cuts.

(7) A concert is organized in the Raksila Hall. The organizers estimate that if the price of a ticket is x euros, then the fans will buy $10000 + 400x - 10x^2$ tickets. The organizers have to choose between two prices: 30 euros per ticket and 40 euros per ticket. Which choice brings more people to the concert? Which choice earns more money for the organizers?

a) 30 euros brings more people and more money.
b) 30 euros brings more people and 40 euros brings more money.

c) 40 euros brings more people and 30 euros brings more money.
d) 40 euros brings more people and more money.

Solution. According to the estimates of the organizers, 30 euro tickets would bring

$$10000 + 400 \cdot 30 - 10 \cdot 30^2 = 10000 + 12000 - 9000 = 13000$$

people to the concert, who would then pay

$$13000 \cdot 30 = 390000 \,\text{euros}$$

for their tickets.

The 40 euro option would bring

$$10000 + 400 \cdot 40 - 10 \cdot 40^2 = 10000 + 16000 - 16000 = 10000$$

people, who would pay

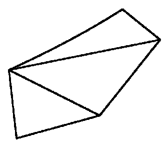
$$10000 \cdot 40 = 400000 \,\text{euros.}$$

We conclude that the 30 euro choice would bring more people and the 40 euro choice would bring more money.

(8) What is the sum of the angles of a pentagon?

a) $480°$ **b)** $540°$ **c)** $600°$ **d)** $720°$

Solution. The sum of the angles of a triangle is $180°$ and a pentagon may be divided into three triangles in such a way that the sum of the angles of the pentagon is the sum of the sums of the angles of the triangles:



Therefore the sum of the angles of a pentagon must be $3 \cdot 180° = 540°$.

(9) The brothers Ibrahim and Hussein were travelling and had just camped alongside the road in order to eat a meal. Ibrahim had prepared five sandwiches and Hussein had prepared three. A stranger showed up and he was also hungry. He asked for food from the brothers and offered to pay eight gold coins for his meal. The brothers agreed and all three ate equal amount of bread. How should the brothers divide the eight gold coins among themselves so that each of them would get the same compensation for each piece of bread they gave to the stranger?

a) Four coins for both.
b) Five coins for Ibrahim and three for Hussein.
c) Six coins for Ibrahim and two for Hussein.
d) Seven coins for Ibrahim and one for Hussein.

Solution. There were 8 sandwiches altogether, and so everyone involved ate $\frac{8}{3}$ sandwiches. Ibrahim gave away $5 - \frac{8}{3} = \frac{15-8}{3} = \frac{7}{3}$ of his sandwiches and Hussein gave away $3 - \frac{8}{3} = \frac{9-8}{3} = \frac{1}{3}$ sandwiches. Therefore Ibrahim gave the stranger seven times as much bread as Hussein did, and so he deserves seven gold coins whereas Hussein deserves one.

(10) Is the number $1 \cdot 3 \cdot 5 + 2 \cdot 4 \cdot 6 + 3 \cdot 5 \cdot 7 + \ldots + 2008 \cdot 2010 \cdot 2012$ even or odd?

a) It is even. **b)** It is odd.

Solution. In the sum every other term is odd and every other term is even. The sum is even or odd depending on whether it has an even or an odd number of odd terms. The first factors of the terms are 1, 2, 3, …, 2008. Thus there are 2008 terms and precisely half of them, 1004 terms, must be odd. The number 1004 is even and so the sum must be even.

(11) Let $X = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$. What can we say about X?

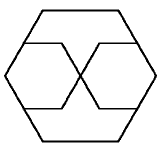
a) $0 < X \leqslant \frac{1}{4}$ **b)** $\frac{1}{4} < X \leqslant \frac{1}{2}$ **c)** $\frac{1}{2} < X \leqslant \frac{3}{4}$ **d)** $\frac{3}{4} < X \leqslant 1$

Solution. Certainly $X > \frac{1}{4}$. On the other hand, we may argue as follows:

$$\begin{aligned}
\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} &< \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64} \\
&= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} \\
&= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \\
&= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\end{aligned}$$

We conclude that $\frac{1}{4} < X \leqslant \frac{1}{2}$.

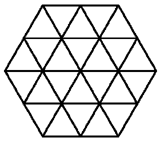
(12) The area of a region having the shape of a regular hexagon is 10. As in the following picture, we remove from it two regions, both of which are of the shape of a regular hexagon with diagonal (from one vertex to the opposing vertex) equal to half of the diagonal (from one vertex to the opposing vertex) of the original hexagon.



What is the area of the remaining region?

a) 3 **b)** 4 **c)** 5 **d)** 6

Solution. We may solve the problem by dividing the entire figure into identical small equilateral triangles:



Now the area of the large hexagon is 24 small triangles and the area of each of the small hexagons is 6 small triangles. The sum of the small hexagons is therefore $6+6=12$ small triangles, which is half of the area of the large hexagon. The area of the remaining region must therefore be one half of the area of the original hexagon, or $\frac{10}{2}=5$.

We could also argue as follows: When the scale is halved, the areas reduce to one fourth. Therefore the area of each of the small hexagons must be precisely one fourth of the area of the large hexagon, and consequently the area of the remaining region must be $1-\frac{1}{4}-\frac{1}{4}=\frac{1}{2}$, or one half, of the area of the original hexagon, which gives $\frac{10}{2}=5$.

(13) How many pairs of integers x, y are there for which $1+x^2=y^2$?

a) 1 **b)** 2 **c)** 4 **d)** more than 4

Solution. The equation says that the squares x^2 and y^2 must be consecutive integers. The squares are $0^2=0$, $1^2=1$, $2^2=4$, $3^2=9$, and so on. It seems that the only squares which are consecutive are 0 and 1. This is actually true and it is easy to see that is so: When n is a positive integer, the difference of the numbers n^2 and $(n+1)^2$ is

$$(n+1)^2-n^2=n^2+2n+1-n^2=2n+1\geqslant 2\cdot 1+1=3>1,$$

and so they can not possibly be consecutive.

The above observation amounts to the fact that the equation $1+x^2=y^2$ holds only when $x^2=0$ and $y^2=1$, or equivalently, when $x=0$ and $y=1$, or $x=0$ and $y=-1$. We conclude that there are precisely two pairs of integers satisfying the equation.

(14) One the angles of a triangle is $72°$ and the difference of its other two angles is $48°$. How large is its largest angle?

a) $72°$ **b)** $78°$ **c)** $82°$ **d)** $88°$

Solution. Let the other two angles of the triangle be x and $x+48°$. Since the sum of the angles of a triangle is $180°$, we must have

$$180°=72°+x+x+48°=120°+2x,$$

from which it follows that $2x = 180° - 120° = 60°$, and also, that $x = 30°$. Now we know that the angles of the triangle are $72°$, $30°$ and $30° + 48° = 78°$. The largest of these is $78°$.