Mathematics Competition for the Seventh Graders of Turku 2012/1/18

Problems and Solutions

(1) Compute $20 \cdot 12 - 11 \cdot 21$.

a) -31 **b)** 0 **c)** 9 **d)** 31

Solution. Direct computation:

$$20 \cdot 12 - 11 \cdot 21 = 240 - 231 = 9.$$

(2) The product of two consecutive natural numbers is 210. How large is the smaller of the numbers?

a) 13 **b)** 14 **c)** 15 **d)** 16

Solution. Since

$$210 = 21 \cdot 10 = 3 \cdot 7 \cdot 2 \cdot 5 = 2 \cdot 7 \cdot 3 \cdot 5 = 14 \cdot 15,$$

the consecutive numbers must be 14 and 15. The smaller of these is 14.

(3) How many rotations does the sweep hand [the hand showing seconds] of an ordinary clock make in one hour?

a) 1 **b)** 12 **c)** 60 **d)** 3600

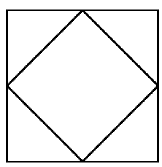
Solution. The sweep hand makes one rotation each minute. There are 60 minutes in an hour, and therefore the sweep hand makes 60 rotations in an hour.

(4) The area of a square is 25. What is its circumference?

a) 5 **b)** 10 **c)** 15 **d)** 20

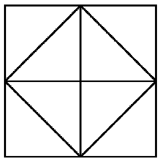
Solution. Since the area of the square is $25 = 5^2$ its side length must be 5. Its circumference is therefore $5 + 5 + 5 + 5 = 20$.

(5) The midpoints of neighboring sides of a $1\,\mathrm{m} \times 1\,\mathrm{m}$-square have been connected with line segments, and we have thereby obtained a smaller square inside the original one. What is the area of the smaller square?



a) $0.25\,\mathrm{m}^2$ **b)** $0.5\,\mathrm{m}^2$ **c)** $1\,\mathrm{m}^2$ **d)** $2\,\mathrm{m}^2$

Solution. Let us divide the figure into identical small triangles as follows:



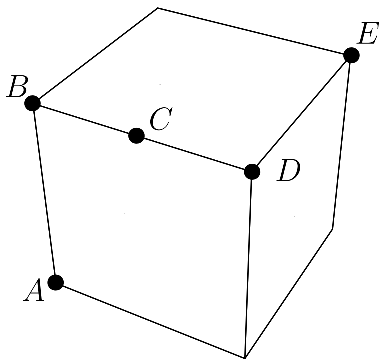
Now the area of the larger square is eight small triangles whereas the area of the smaller square is four small triangles. Thus the area of the smaller square is one half of the area of the larger square, or $0.5\,\mathrm{m}^2$.

(6) In order to build a small forest cabin, we need one hundred logs, each of which must be five meters long. In the beginning, we only have logs which are twenty meters long each. What is the smallest number of times we have to saw through a log in order to obtain the five meter logs we need?

a) 50 **b)** 75 **c)** 99 **d)** 100

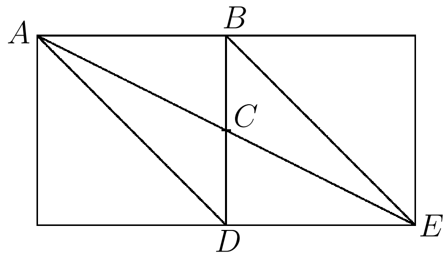
Solution. By cutting three times we obtain four five meter logs from one twenty meter log. Therefore we obtain $100 = 25 \cdot 4$ five meter logs by making $25 \cdot 3 = 75$ cuts.

(7) An ant may crawl across the surface of a cube every which way it wants. In the beginning it is at the vertex A and it would like to get to the vertex E. Which of the points B, C and D lies on a shortest possible path?



a) The point B. **b)** The point C. **c)** The point D.

Solution. Let us cut off the two faces touching the point C and let us unfold them on a plane. This does not change the distances involved.



But now the shortest path between the points A and E is a straight line which passes through the point C.

(8) A concert is organized in the Turku Hall. The organizers estimate that if the price of a ticket is x euros, then the fans will buy $10000 + 400x - 10x^2$ tickets. The organizers have to choose between two prices: 30 euros per ticket and 40 euros per ticket. Which choice brings more people to the concert? Which choice earns more money for the organizers?

a) 30 euros brings more people and more money.
b) 30 euros brings more people and 40 euros brings more money.
c) 40 euros brings more people and 30 euros brings more money.
d) 40 euros brings more people and more money.

Solution. According to the estimates of the organizers, 30 euro tickets would bring

$$10000 + 400 \cdot 30 - 10 \cdot 30^2 = 10000 + 12000 - 9000 = 13000$$

people to the concert, who would then pay

$$13000 \cdot 30 = 390000 \, \text{euros}$$

for their tickets.

The 40 euro option would bring

$$10000 + 400 \cdot 40 - 10 \cdot 40^2 = 10000 + 16000 - 16000 = 10000$$

people, who would pay

$$10000 \cdot 40 = 400000 \, \text{euros}.$$

We conclude that the 30 euro choice would bring more people and the 40 euro choice would bring more money.

(9) With how many zeroes does the number $1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 30$ end?

a) 4 **b)** 5 **c)** 6 **d)** 7

Solution. The relevant issue here is how many times does the number ten divide the product, or equivalently, how many times do the numbers two and five divide it?

The number two appears in the multiplicands $2, 4, 6, \ldots, 30$ and therefore divides the product at least 15 times.

The number five divides only the multiplicands 5, 10, 15, 20, 25 and 30, and appears in each of these as a factor only once, except for the number 25 in which it appears twice. Thus the number five divides the product in question precisely $1 + 1 + 1 + 1 + 2 + 1 = 7$ times.

We arrive at the conclusion that the number 10 divides the product precisely 7 times, so that the product ends with exactly seven zeroes.

(10) Let $X = 1 + 2 + 3 + \ldots + 63 + 64 + 65 + 64 + 63 + \ldots + 3 + 2 + 1$. How large is X?

a) 2015 **b)** 2080 **c)** 4180 **d)** 4225

Solution. The number X is the sum of the numbers

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & \cdots & 63 & 64 & 65 \\ 64 & 63 & 62 & 61 & \cdots & 2 & 1 & \end{array}$$

But in this array the sum of the numbers of each column is 65, and there are 65 columns. Therefore $X = 65 \cdot 65 = 4225$.

(11) The boys living in a six-storey building wanted to find out which is the tallest fall to the lawn a bottle survives. They had two identical bottles at their disposal. They figured out that if a bottle breaks when dropped from a certain floor of the building, then it would have broken also if it had been dropped from any higher floor. The boys wanted to investigate the matter in fewest possible experiments, as the risk of the irritable janitor finding out about the research kept constantly rising. From which floor should they drop a bottle first?

a) The first floor. **b)** The third floor. **c)** The fourth floor. **d)** The sixth floor.

Solution. There are seven possible answers to the question of the boys. (After all, it is also possible that the bottle breaks even when dropped only from the height of the first floor.) Two experiments can not possibly clear the matter, as there are only four possible ways the two experiments could go. Therefore the boys must be prepared to drop a bottle three times.

Let us assume, that we can clear the matter dropping a bottle only three times. It is clear that the first drop should not be done from the first floor, for if the bottle survived that fall, we would have to find out the answer to the question among six different consecutive possibilities with merely two experiments. Similarly, the first experiment should not be done from the sixth floor, for if the bottle broke in that fall, we would have to find the right answer among six consecutive ones with only two experiments.

It is also not sufficient to drop a bottle from the fourth floor. Let us assume that we did so and the bottle broke. Then we would have to decide between four consecutive possibilities with only two experiments. But now our two experiments only have three possible outcomes as we only had two bottles at our disposal, since if the second bottle also broke we could no more do a third drop since all of our bottles would be broken.

In the same spirit, we may show that the first experiment should not be done from the second or fifth floors.

Finally, it is easy to see that by dropping the first bottle from the third floor, we may answer the question of the boys by doing only three experiments. The second test should be done from the first or from the fifth floor depending on what happens to the bottle in the first fall.

(12) Let $X = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729}$. What can we say about X?

a) $0 < X \leqslant \frac{1}{4}$ **b)** $\frac{1}{4} < X \leqslant \frac{1}{2}$ **c)** $\frac{1}{2} < X \leqslant \frac{3}{4}$ **d)** $\frac{3}{4} < X \leqslant 1$

Solution. Certainly $X > \frac{1}{4}$, for $\frac{1}{3} > \frac{1}{4}$. On the other hand, since all the denominators are powers of three, we have

$$X = \frac{243 + 81 + 27 + 9 + 1}{729} = \frac{361}{729},$$

and $2 \cdot 361 = 722 < 729$, so that $X < \frac{1}{2}$.

We have arrived at the conclusion that $\frac{1}{4} < X \leqslant \frac{1}{2}$.

(13) The sum of the vertex angle and a base angle of an isosceles triangle is $112°$. How large is the vertex angle?

a) $24°$ **b)** $34°$ **c)** $44°$ **d)** $54°$

Solution. When we add the remaining base angle to the sum $112°$ we obtain the sum of the angles of the triangle, which is $180°$. Therefore a base angle of the triangle in question must be $180° - 112° = 68°$. Now the sum of the base angles is $68° + 68° = 136°$ and the vertex angle must be $180° - 136° = 44°$.

(14) Are there integers x and y so that $x^2 + 6 = y^2$?

a) Yes, there are. **b)** No, there are not.

Solution. The sequence of square numbers begins as follows:

$$0, \quad 1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad \ldots$$

The sequence of the differences of consecutive squares is

$$1, \quad 3, \quad 5, \quad 7, \quad 9, \quad 11, \quad 13, \quad 15, \quad \ldots,$$

which gives rise to the suspicion that from the square $3^2 = 9$ onwards the consecutive squares are more than six units apart from each other. This is actually so. Namely, if $n \geqslant 3$ is a natural number, then

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1 \geqslant 2 \cdot 3 + 1 = 7 > 6.$$

The above observations mean that if we had $x^2+6 = y^2$, then we would necessarily have $x^2 \leqslant 2^2 = 4$. But since the numbers

$$0^2 + 6 = 6, \quad 1^2 + 6 = 7, \quad \text{ja} \quad 2^2 + 6 = 10$$

are not squares, we conclude that there are no integers x and y satisfying the given equation.