Mathematics Competition for the Seventh graders of Oulu Sub-Region, Final 17.4.2021
Solutions

1. Jussi has two ten litre buckets and one three litre bucket. There is nine litres of water in one of the big buckets and the other is exactly half full. The three litre bucket is empty.

Jussi wants to measure exactly seven litres of water in each of the big buckets and keep the small bucket empty. How can Jussi do this? He can pour water from a bucket to another but, upon doing so, has to keep on pouring until either one of the buckets is full or empty. Jussi can't get more water anywhere and he has no other tools available.

Solution. Let us denote the ten litre bucket with 9 litres of water by A, the three litre bucket by B and the other ten litre bucket with 5 litres of water by C.

We first pour water from bucket A to bucket C so that bucket A has 4 litres of water and bucket C is full. After this we pour water from bucket C to bucket B so that bucket B is full. Then bucket C has 7 litres of water in it. Lastly, we pour the water from bucket B to bucket A leaving bucket B empty and bucket A with 7 litres of water.

2. You find a machine that has five levers in a row. The lengths of the levers from left to right are $3, 2, 1, 2$ and 3. By pulling all of the levers in correct order you open a hidden door and find a treasure. On your previous adventures you have collected the following clues to help you determine the correct order for pulling the levers:

1) The first lever has no longer lever next to it.
2) The second lever is on the right side of the shortest lever.
3) The third lever is shorter than any of the levers pulled so far.
4) The fourth lever is not the shortest nor the longest lever.
5) The last lever has a lever on both sides of it and one of them is the lever that was pulled second.

What is the correct order for pulling the levers to open the hidden door?

Solution. Let us denote the levers from left to right by A, B, C, D and E respectively. Deducing from the clues, we write down the possibilities for levers pulled in each stage in the table below.

Clue	A	B	C	D	E
1	x				x
2				x	x
3		x	x	x	
4		x		x	
5			x	x	

We can see from the table that the first lever to be pulled must be A since the only clue it satisfies is the first clue. Then the only option left for lever E is to be pulled second. Because lever E is pulled second, the last lever to be pulled must be D. Hence lever C is the third lever to be pulled. The fourth lever to be pulled is then lever B.

3.

a) Determine and justify whether the following claim is true or false: Whenever two polygons have the same perimeter then they also have the same area.

b) Give an example of such a rectangle that the numeric value of its perimeter equals the numeric value of its area.

Solution. a) The claim is false: For example, a square with side length 2 has perimeter $4 \cdot 2 = 8$ and area $2 \cdot 2 = 4$. On the other hand, a rectangle with side lengths 1 and 3 has also perimeter $8 = 2 \cdot 1 + 2 \cdot 3$, but the area of the rectangle is $3 \cdot 1 = 3 \neq 4$.

b) For example, a square with side length 4 has perimeter $4 \cdot 4 = 16$ and its area is also $4 \cdot 4 = 16$.

4. Determine all integers x and y that satisfy the equation

$$xy + x = 23.$$

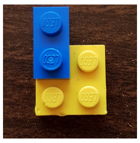
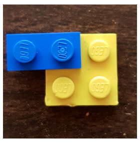
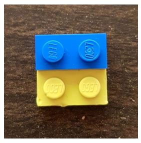
Solution. The equation can be simplified as

$$x(y+1) = 23.$$

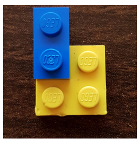
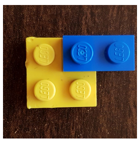
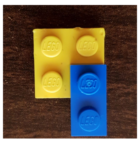
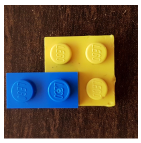
Since 23 is a prime number, there are only two ways to factor it: $23 = 1 \cdot 23 = (-1) \cdot (-23)$. Hence the only integer solutions to the equation are

$$\begin{aligned}
x &= 1 \text{ and } y+1 = 23 \text{ that is } y = 22,\\
x &= 23 \text{ and } y+1 = 1 \text{ that is } y = 0,\\
x &= -1 \text{ and } y+1 = -23 \text{ that is } y = -24,\\
x &= -23 \text{ and } y+1 = -1 \text{ that is } y = -2.
\end{aligned}$$

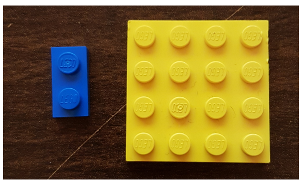
5. The blue 1×2 Lego brick can be attached on top of the yellow 2×2 Lego brick in three different ways:

When counting the number of different ways to attach the bricks, the cases where one way can be attained by rotating another way are counted as one. For example all four ways shown below are considered as same when counting the ways to attach the bricks:

How many different ways are there to attach the blue 1×2 Lego brick on top of the yellow 4×4 Lego brick when the ways attained by rotation are considered the same?



Solution. If the blue brick is attached horizontally the left stud of the blue brick can be attached to any of the 16 studs of the yellow brick. There are also 4 ways to attach the blue brick so that the left stud is not on the yellow brick. Hence there are 20 ways to attach the blue brick horizontally on top of the yellow brick. Similarily there are 20 ways to attach the blue brick vertically on top of the yellow brick. Without taking rotations into account, there are 40 ways in total to attach the blue brick on top of the yellow brick. No matter how the blue brick is attached, we get a different position for it when the whole thing is rotated by $90°$, $180°$ or $270°$. Hence each way to attach the blue brick is calculated four times in the calculation above. Taking the rotations into account, there are then $40/4 = 10$ ways to attach the blue brick on top of the yellow brick.